- The divergence thm:

For a function $u: [a,b] \to \mathbb{R}$, we know that the following fundamental theorem of calculus holds:

$$\int_a^b u'(x)\,dx = u(b) - u(a),$$

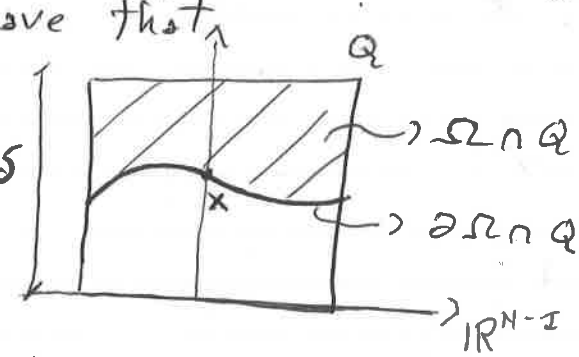
providing $u \in C^1([a,b])$.

The above theorem relates the integral of a derivative with the value of the function at the boundary points of the set we are integrating on.

We would like to state an analogous of the above theorem for functions $u: \mathbb{R}^N \to \mathbb{R}$, and its partial derivatives. First of all we need to specify the kind of sets we consider admissible to integrate on.

- Def.: let $\Omega \subseteq \mathbb{R}^N$ be an open, bounded set. We say that Ω is a set of class C^1 if, for every $x \in \partial\Omega$ there exist a square Q centered in x with the following property:

  up to rotating and translating Q, we have that

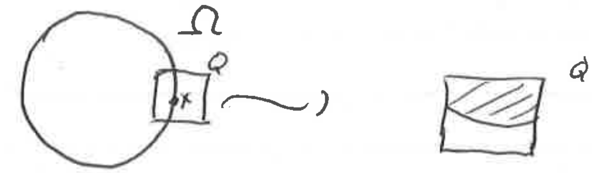


$$\Omega \cap Q = \{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N \in (\varphi(x'), \delta) \},$$
$$\partial\Omega \cap Q = \{ (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : x_N = \varphi(x') \},$$

where $\varphi: \mathbb{R}^{N-1} \to \mathbb{R}$ is a function of class C^1.

· <u>Remark</u>: what we asking for is the boundary of the set Ω to be locally representable by a C^1 function (that, of course, can change from point to point on the boundary!) and that has Ω only on one side of it.

· <u>Examples</u>: i)



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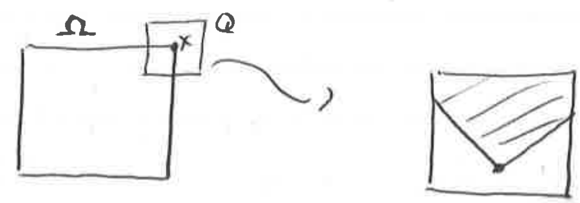
ii)



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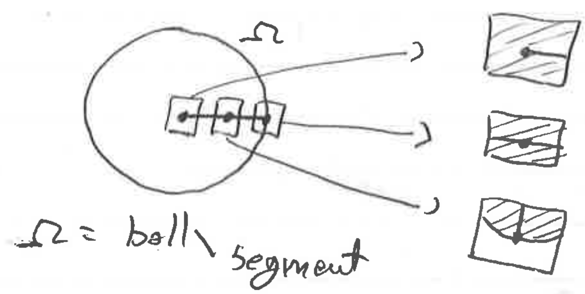
iii)



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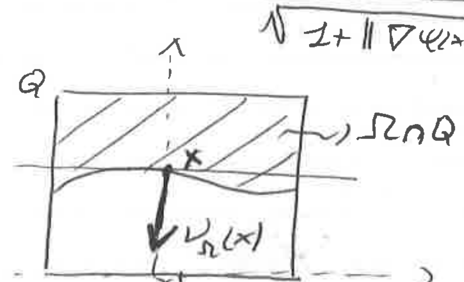
→ points for which the definition does not apply, are called <u>singular</u> points.

· <u>Def.</u>: let $\Omega \subseteq \mathbb{R}^N$ be an open bounded set of class C^1. For every $x \in \partial\Omega$ we define the <u>exterior normal to Ω at x</u>, as the vector $\nu_\Omega(x)$ defined as:

$$\nu_\Omega(x) := \frac{(\nabla\varphi(x'), -1)}{\sqrt{1 + \|\nabla\varphi(x')\|^2}},$$



where: $x = (x', x_N)$, and φ is any function $\mathbb{R}^{N-1}$ representing $\partial\Omega$ and Ω locally; as

- <u>Remark</u>: - the definition does <u>not</u> depend on the choice of the function φ
  - the vector $\nu_\Omega(x)$ points outside Ω, in the normal direction to $\partial\Omega$.
  [that is, the hyperplane $\nu_\Omega(x)^\perp$, of vectors v that are orthogonal to $\nu_\Omega(x)$, is tangent to $\partial\Omega$ at x].

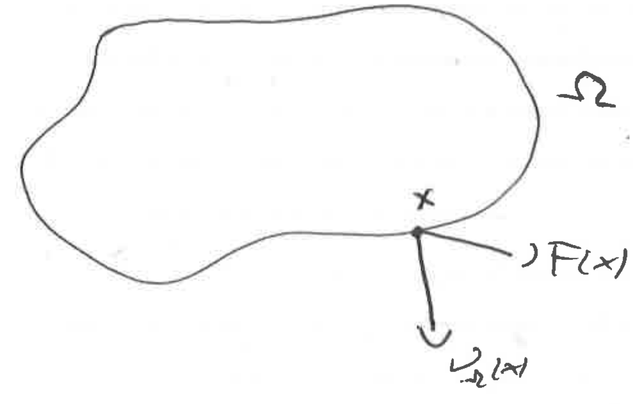
The generalization of the fundamental theorem of calculus to higher dimensions, is called the <u>divergence theorem</u> [or the <u>Gauss-Ostrogradskij thm</u>]:

- <u>Thm</u>: (<u>divergence thm</u>)
  Let $F: \mathbb{R}^n \to \mathbb{R}^n$ be a vector field of class C^1;
  i.e., $F = (F_1, \dots, F_n)$ and every component $F_i: \mathbb{R}^n \to \mathbb{R}$ is a function of class C^1.
  Let $\Omega \subseteq \mathbb{R}^n$ be an open bounded set of class C^1.
  Then:

$$\boxed{\int_\Omega \operatorname{div} F(x)\,dx = \int_{\partial\Omega} F(y)\cdot\nu_\Omega(y)\,ds(y),}$$

where ds is the measure on the surface.

→ The theorem relates the integral of the divergence of the vector field F with the <u>flux</u> $[F\cdot\nu_\Omega]$ of F on the surface $\partial\Omega$.



We will prove the theorem only in dimension $N=2$. In this case, the boundary of a set of class C^1 is a C^1 curve. Thus, we need to specify what we mean by the measure on the curve.

To justify our definition, we reason as follows: assume we want to give a meaning to the integral of a function

$$u : \mathbb{R}^2 \to \mathbb{R}$$

over the curve

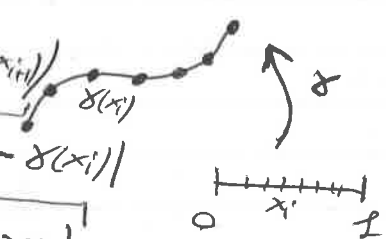
$$\gamma : \mathbb{R} \to \mathbb{R}^2$$
$$s \mapsto (\gamma_1(s), \gamma_2(s))$$

[or better, the image of the curve γ]

$$\boxed{\int_\gamma u}$$

To be more precise, let $\gamma : [0,1] \to \mathbb{R}^2$ be a curve of class C^1, and let $u : \mathbb{R}^2 \to \mathbb{R}$.
By mimic the Riemann sum, we can say that

$$\int_\gamma u \sim \sum_i u(\gamma(x_i)) \underbrace{\text{length}(\gamma(x_i), \gamma(x_{i+1}))}_{\sim |\gamma(x_{i+1}) - \gamma(x_i)|}$$



$$\sim \underbrace{\sum_i u(\gamma(x_i)) \overbrace{|\gamma'(x_i)|\,(x_{i+1} - x_i)}}$$

the curve γ is of class C^1

this is the Riemann sum of

$$\int_0^1 u(\gamma(x))\, |\gamma'(x)|\, dx$$

Thus, the following definition feels natural now:

- **Def.:** let $\gamma : [0,1] \to \mathbb{R}^2$ be a curve of class C^1, and let $u : \mathbb{R}^2 \to \mathbb{R}$.
We define:

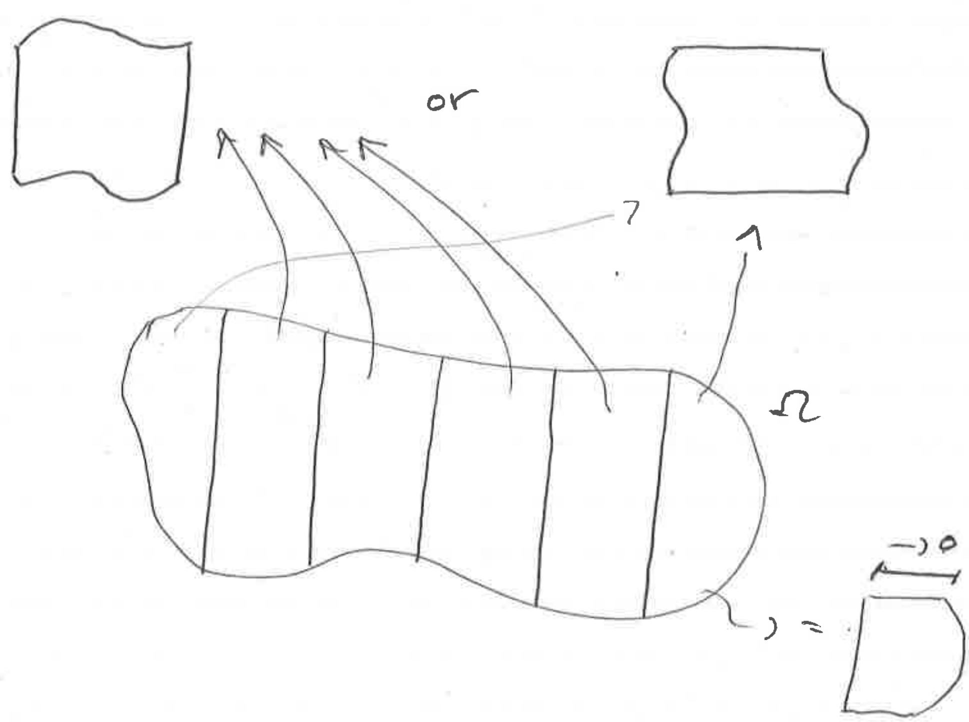
$$\int_\gamma u := \int_0^1 u(\gamma(x))\,|\gamma'(x)|\,dx.$$

- Remark: the above definition depends only on $\gamma([0,1])$, not on γ!

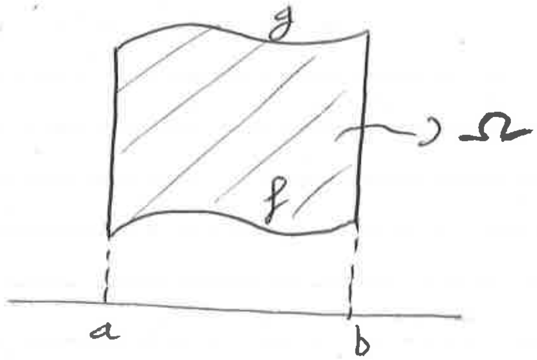
The divergence thm in dimension $N=2$ translates as:

$$\int_\Omega \operatorname{div} F(x)\,dx = \int_{\partial\Omega} F \cdot \nu_{\partial\Omega},$$

where the second integral has now a precise meaning, since $\partial\Omega$ is a C^1 curve on $\mathbb{R}^2$.
In order to prove the above equality, we first notice that a set $\Omega \subseteq \mathbb{R}^2$ of class C^1 can be seen as union of sets of the form:



Thus, it is enough to prove the theorem in the case Ω is of one of the two types above.
Since the computations are basically the same, we focus on the first kind:



So, assume

$$\Omega = \{ (x,y) \in \mathbb{R}^2 : x \in (a,b), \ f(x) < y < g(x) \},$$

where $f, g : [a,b] \to \mathbb{R}$ are C^1 functions, $f \leq g$.
Now, let us take a vector field $F : \mathbb{R}^2 \to \mathbb{R}^2$ of class C^1. We have that:

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}, \qquad F = (F_1, F_2).$$

So, we have that:

- $$\int_\Omega \frac{\partial F_2}{\partial y} \, dx\,dy = \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial F_2}{\partial y}(x,y)\, dy \right] dx$$

by using Fubini's thm

$$= \int_a^b \left[F_2(x, g(x)) - F_2(x, f(x)) \right] dx.$$

by the fundamental thm of calculus

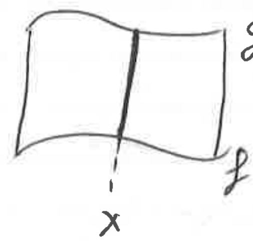
$$\bullet \int_{\Omega} \frac{\partial F_2}{\partial x}\, dx\,dy = \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial F_2}{\partial x}(x,y)\, dy \right] dx.$$

Now, consider the function:

$$A : [a,b] \to \mathbb{R}$$

defined as:

$$A(x) := \int_{f(x)}^{g(x)} F_2(x,y)\, dy.$$



We have that:

$$A'(x) = F_2(g(x),y)\, g'(x) - F_2(f(x),y)\, f'(x) + \int_{f(x)}^{g(x)} \frac{\partial F_2}{\partial x}\, dy.$$

Thus,

$$\int_{\Omega} \frac{\partial F_2}{\partial x}\, dx\,dy = \int_a^b \left[\int_{f(x)}^{g(x)} \frac{\partial F_2}{\partial x}(x,y)\, dy \right] dx$$

$$= \int_a^b \left[A'(x) - F_2(x, g(x))\, g'(x) + F_2(x, f(x))\, f'(x) \right] dx$$

$$= A(b) - A(a) - \int_a^b \left[F_2(x, g(x))\, g'(x) - F_2(x, f(x))\, f'(x) \right] dx$$

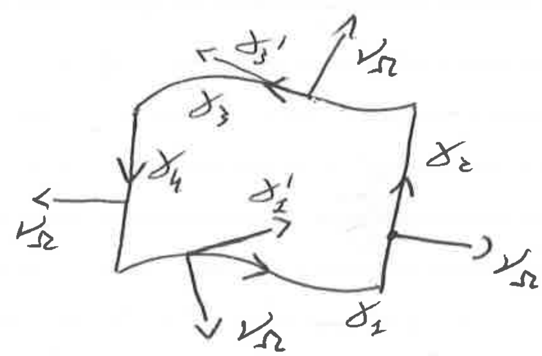
$$= \int_{f(b)}^{g(b)} F_2(b,y)\, dy - \int_{f(a)}^{g(a)} F_2(a,y)\, dy$$

$$- \int_a^b \left[F_2(x, g(x))\, g'(x) - F_2(x, f(x))\, f'(x) \right] dx$$

We now have to interpret all the integrals we got as integrals over the curve $\partial\Omega$. For, let us see $\partial\Omega$ as the curve γ given by the union of the following curves:

- $\gamma_1 : [a,b] \longrightarrow \mathbb{R}^2;\quad \gamma_1(x) := (x, f(x)),$
- $\gamma_2 : [g(a), g(b)] \longrightarrow \mathbb{R}^2;\quad \gamma_2(s) := (b, s),$
- $\gamma_3 : [a,b] \longrightarrow \mathbb{R}^2;\quad \gamma_3(x) := (b+a-x, g(b+a-x)),$
- $\gamma_4 : [f(a), g(a)] \longrightarrow \mathbb{R}^2;\quad \gamma_4(s) := (a, f(a)+g(a)-s).$

- Remark: strictly speaking, we have a problem of regularity in the points where the curves meet. But they are only four points, so we can forget about them!



Moreover, we have that:

- $z \in \gamma_1 : \quad \gamma_1'(z) = \dfrac{(1, f'(z))}{|\gamma_1'(z)|} \rightsquigarrow \nu_\Omega(z) = \dfrac{(f'(z), -1)}{|\gamma_1'(z)|},$
- $z \in \gamma_2 : \quad \gamma_2'(z) = (0, 1) \rightsquigarrow \nu_\Omega(z) = (1, 0),$
- $z \in \gamma_3 : \quad \gamma_3'(z) = \dfrac{(-1, -g'(z))}{|\gamma_3'(z)|} \rightsquigarrow \nu_\Omega(z) = \dfrac{(-g'(z), 1)}{|\gamma_3'(z)|},$
- $z \in \gamma_4 : \quad \gamma_4'(z) = (0, -1) \rightsquigarrow \nu_\Omega(z) = (-1, 0).$

$\longrightarrow \nu_\Omega(z) = (\nu^1(z), \nu^2(z))$

Thus, we have that:

- $$\int_a^b F_2(x,g(x))\,dx = \int_a^b F_2(x,g(x))\,\nu_\Omega^2(x,g(x))\,|\gamma_3'(x,g(x))|\,dx$$
$$= \int_{\gamma_3} F_2 \nu_\Omega^2$$

- $$\int_a^b -F_2(x,f(x))\,dx = \int_a^b F_2(x,f(x))\,\nu_\Omega^2(x,f(x))\,|\gamma_1'(x,f(x))|\,dx$$
$$= \int_{\gamma_1} F_2 \nu_\Omega^2$$

- $$\int_{f(b)}^{g(b)} F_1(b,y)\,dy = \int_{f(b)}^{g(b)} F_1(b,y)\,\nu_\Omega^1(b,y)\,dy = \int_{\gamma_2} F_1 \nu_\Omega^1$$

- $$\int_{f(a)}^{g(a)} -F_1(a,y)\,dy = \int_{f(a)}^{g(a)} F_1(a,y)\,\nu_\Omega^1(a,y)\,dy = \int_{\gamma_4} F_1 \nu_\Omega^1$$

- $$\int_a^b -F_1(x,g(x))\,g'(x)\,dx = \int_a^b F_1(x,g(x))\,\nu_\Omega^1(x,g(x))\,|\gamma_3'(x,g(x))|\,dx$$
$$= \int_{\gamma_3} F_1 \nu_\Omega^1$$

- $$\int_a^b F_1(x,f(x))\,f'(x)\,dx = \int_a^b F_1(x,f(x))\,\nu_\Omega^1(x,f(x))\,|\gamma_1'(x)|\,dx$$
$$= \int_{\gamma_1} F_1 \nu_\Omega^1$$

Thus, we have that:

$$\int_\Omega \operatorname{div} F \, dx\,dy = \int_{\gamma_1} (F_1 \nu_\Omega^1 + F_2 \nu_\Omega^2) + \int_{\gamma_2} F_1 \nu_\Omega^1 + \int_{\gamma_3} (F_1 \nu_\Omega^1 + F_2 \nu_\Omega^2) + \int_{\gamma_4} F_1 \nu_\Omega^1$$

$$= \int_{\partial\Omega} F \cdot \nu_\Omega .$$

This proves the divergence theorem in $N=2$.
We now want to see some consequences of this result.
If we take the vector field

$$F : \mathbb{R}^2 \to \mathbb{R}^2$$

given by:

$$F(x,y) = (u(x,y), 0),$$

where $u : \mathbb{R}^2 \to \mathbb{R}$ is a C^1 function, we get:

$$\int_\Omega \operatorname{div} F \, dx\,dy = \int_{\partial\Omega} F \cdot \nu_\Omega$$

$$\boxed{\int_\Omega \frac{\partial u}{\partial x} \, dx\,dy = \int_{\partial\Omega} u \, \nu_\Omega^1}$$

Similarly, by taking: $F(x,y) := (0, u(x,y))$, we get

$$\boxed{\int_\Omega \frac{\partial u}{\partial y} \, dx\,dy = \int_{\partial\Omega} u \, \nu_\Omega^2}$$

Another important corollary is the following:
let $u: \mathbb{R}^2 \to \mathbb{R}$ be a C^2 function, and consider the vector field given by

$$\nabla u : \mathbb{R}^2 \to \mathbb{R}^2 \\ (x,y) \mapsto \nabla u(x,y).$$

By the divergence theorem, we get:

$$\int_\Omega \operatorname{div}(\nabla u)\, dx\, dy = \int_{\partial\Omega} \nabla u \cdot \nu_\Omega .$$

But:

- $\operatorname{div}(\nabla u) = \Delta u$
- $\nabla u \cdot \nu_\Omega = \dfrac{\partial u}{\partial \nu_\Omega} =: \partial_{\nu_\Omega} u.$

Thus, we get:

$$\boxed{\int_\Omega \Delta u \, dx\, dy = \int_{\partial\Omega} \partial_{\nu_\Omega} u}$$

Finally, we want to recover the so called
<u>Gauss-Greens identities</u>:

$$\int_\Omega (u \Delta v + \nabla u \cdot \nabla v)\, dx = \int_{\partial\Omega} u\, \partial_\nu v \, ds$$

$$\int_\Omega (u \Delta v - v \Delta u)\, dx = \int_{\partial\Omega} (u \partial_\nu v - v \partial_\nu u)\, ds$$

where $u, v: \mathbb{R}^2 \to \mathbb{R}$ are functions of class C^2,
and, for simplicity, we denoted $\nu_\Omega =: \nu$.

Just notice that:

- $$\operatorname{div}(u \nabla v) = u \Delta v + \nabla u \cdot \nabla v$$

$$\Downarrow$$

$$\int_\Omega (u \Delta v + \nabla u \cdot \nabla v)\, dx = \int_\Omega \operatorname{div}(u \nabla v)\, dx$$

$$= \int_{\partial\Omega} (u \nabla v) \cdot \nu \, ds = \int_{\partial\Omega} u \, \partial_\nu v \, ds$$

- notice that:

$$\int_\Omega (u \Delta v - v \Delta u)\, dx =$$

$$= \int_\Omega (u \Delta v + \nabla u \cdot \nabla v)\, dx - \int_\Omega (v \Delta u + \nabla v \cdot \nabla u)\, dx$$

$$= \int_{\partial\Omega} u \, \partial_\nu v \, ds - \int_{\partial\Omega} v \, \partial_\nu u \, ds.$$